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On A Palm Probabilistic Approach for the Comparison of Cohort Fertility

The Object and the Methodology

THE present note envisages a new application of palm probability for the comparison of fertility status of two consecutive marriage cohorts; (the mean difference in the age of the two marriage cohorts being / years say) based on incomplete fertility history of women upto I-th order of birth for the elderly cohort; and upto (i — l)-th order of birth for the younger cohort till the point of observation (i = 1, 2, 3, . . .).

The open birth interval probability distribution of two cohorts following the i-th and *(i* — l)-th order of births respectively have been obtained by using William Brass Model with parameters *(k, a)* for the elderly cohort and *(k', a)* for the younger cohort. The estimates of *(k, a)* and *(k', a')* have been obtained by the method of moments, utilizing the fertility history of both the cohorts upto the point of observation; alongwith some plausible assumption relating to the pregnancy rate of a cohort observed upto a fixed period of time with that of the open pregnancy rate following the same period (i.e., up to the end of the fertility span or reproductive period).

Development of the Model

(a) $\{V_r(t) \quad i\}$ = Conditional probability of r-birihs in a time period *t* given that i'-th birth (i -= 1, 2, 3, . . .) have taken place precisely at period $T = t$. This is known as *palm probability.*

(b) $\ddot{o}_i^{\{i\}}(t)$ = probability of r births in time *t* for mothers having *i* births upto the point of observation.

(c) $\lambda^{(i)}$ mean pregnancy rate of mothers following the i-th pregnancy upto the end of the reproductive period and it is a Poisson parameter.

(d) We assume $\lambda^{(i)}$ varying from person to person assumes a two parameter family of Γ distribution given by

$$f(\lambda^{(i)}) = \frac{a^k \, e^{-a\lambda^{(i)}} \, (\lambda^{(i)})^k}{\Gamma(k)}; \quad 0 \leqslant \lambda^{(i)} < \infty \; k, a > 0$$

$$f(\lambda^{(i-1)}) = \frac{a'^{k'} \, e^{-a'\lambda^{(i-1)}} \, (\lambda^{(i-1)})^{k'}}{\Gamma(k')}; \; 0 \leqslant \lambda^{(i-1)} < \infty \; k', a' > 0.$$

(e) $\{\lambda^{(0)} \mid i\}$ represents the mean pregnancy rate from the date of the marriage upto i-th order of parity

obviously, $$E(\lambda^{(i)}) = \frac{k}{a}; \quad \text{Var}(\lambda^{(i)}) = \frac{k}{a^2};$$

$$E[(\lambda^{(i)})^3] = \frac{k(k+1)(k+2)}{a^3};$$

Similar moments are obtainable for $\lambda^{(i-1)}$; k, a being replaced by k', a' respectively.

(f) William Brass Model (1958) applied for the probability distribution of births during a period t with fertility parameters (k, a) applicable when i number of births have already taken place upto the point of observation $T = t$. This is given by

$$\phi_r^{(i)}(t) = \frac{\Gamma(k+r)}{\Gamma(k)\, r!} \left(\frac{a}{a+t}\right)^k \left(\frac{t}{a+t}\right)^r.$$

Then the palm probability $\{V_r(t)/i\}$ $i = 1, 2, 3, \ldots$ is given by Khintchine (1960) as

$$\{V_r(t)/i\} = \frac{k}{a}\int_0^t \{\phi_{r-1}(u) - \phi_r(u)\}\, du$$

$$= \frac{k}{a}\left[\int_0^t \left\{\frac{\Gamma(k+r-1)}{\Gamma(k)(r-1)!}\left(\frac{a}{a+u}\right)^k \left(\frac{u}{a+u}\right)^{r-1} - \frac{\Gamma(k+r)}{\Gamma(k)\, r!}\left(\frac{a}{a+u}\right)^k \left(\frac{u}{a+u}\right)^r\right\} du\right].$$

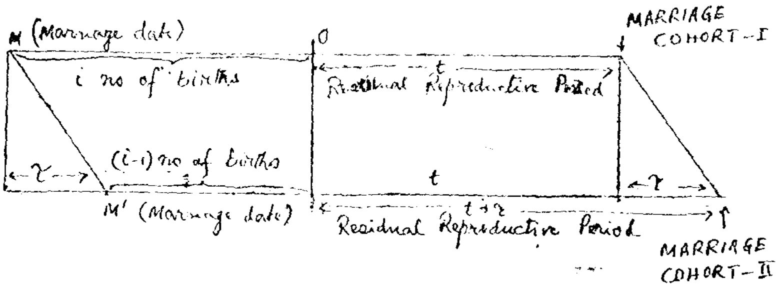
Applying direct Integration gives for $r = 1, 2, 3, 4$ the palm probabilities of additional births till a further period of time $T = t$ are obtainable as

$$\{V_1 (t)/i\}_{k,a} = k \left(\frac{a}{a+t}\right)^{k-1} \left(\frac{t}{a+t}\right)$$

$$\{V_2 (t)/i\}_{k,a} = \frac{k^2}{2} \left(\frac{a}{a+t}\right)^{k-1} \left(\frac{t}{a+t}\right)^2$$

$$\{V_3 (t)/i\}_{k,a} = \frac{k^2 (k+1)}{6} \left(\frac{a}{a+t}\right)^{k-1} \left(\frac{t}{a+t}\right)^3$$

$$\{V_4 (t)/i\}_{k,a} = \frac{k^2 (k+1)(k+2)}{6} \left(\frac{a}{a+t}\right)^{k-1} \left(\frac{t}{a+t}\right)^4$$



and finally for r number of births

$$\{V_r (t)/i\} = \frac{k^2 (k+1) \ldots (k+r-1)}{r\,!} \left(\frac{a}{a+t}\right)^{k-1} \left(\frac{t}{a+t}\right)^r$$

$$= \frac{k\,\Gamma (k+r-1)}{\Gamma (k)\, r\,!} \left(\frac{a}{a+t}\right)^{k-1} \left(\frac{t}{a+t}\right)^r \quad (2)$$

$$E (r/i) = \sum_{r=1}^{N_i} \frac{k\,\Gamma (k+r-1)}{\Gamma (k)\, r\,!}\, r \left(\frac{a}{a+t}\right)^{k-1} \left(\frac{t}{a+t}\right)^r . \quad (3)$$

Where N_i represents the maximum possible number of births following i number of births in the entire residual reproductive span.

$$E (r/i) = \frac{k}{\Gamma (k)} \left(\frac{a}{a+t}\right)^{k-1} \sum_{r=1}^{N_i} \frac{\Gamma (k+r-1)}{(r-1)\,!} \left(\frac{t}{a+t}\right)^r .$$

Now $$\frac{\Gamma(k+r-1)}{\Gamma(r)} \simeq \frac{\Gamma(r)+(k-1)\Gamma'(r)+O(\Gamma''(r))}{\Gamma(r)}$$

$$\simeq [1+(k-1)\Gamma'(r)/\Gamma(r)] \qquad (3')$$

using approximation for large r we have

$$\frac{d}{dr}(\log\Gamma(r)) = \log r - \frac{1}{2r} + O\left(\frac{1}{r^2}\right) \qquad (3'')$$

on substitution of (3″) in (3′)

$$E(r/i) = \frac{k}{\Gamma(k)}\left(\frac{a}{a+t}\right)^{k-1}\left\{\sum_{r=1}^{N_i}\left(\frac{t}{a+t}\right)^r\right.$$

$$\left. + (k-1)\left[\sum_{r=1}^{N_i}\log r\left(\frac{t}{a+t}\right)^r - (k-1)\sum_{r=1}^{N_i}\frac{1}{r}\left(\frac{t}{a+t}\right)^r\right]\right\}$$

$$\cong \frac{k}{\Gamma(k)}\left(\frac{a}{a+t}\right)^{k-1}\left\{\left[\frac{t}{a+t}\cdot\frac{1-(t/(a+t))^{N_i}}{[1-(t/a+t)]}\right]\right.$$

$$\left. + (k-1)\sum_{r=1}^{N_i}\log r\left(\frac{t}{a+t}\right)^r - \frac{(k-1)}{2}\left[-\log_e\left(1-\frac{t}{a+t}\right)\right]\right\}. \qquad (3\ \ )$$

The second term within parenthesis on the R.H.S. of (3″) ⇒

$$\sum_{r=1}^{N_i}\log r\left(\frac{t}{a+t}\right)^r = \left\{\Delta^{-1}\left[\left(\frac{t}{a+t}\right)^r\log r\right]\right\}_1^{Ni+1}$$

$$= \left\{(E-1)^{-1}\left(\frac{t}{a+t}\right)^r\log r\right\}_1^{Ni+1} \qquad (4)$$

Using $\phi(E)[\alpha^r f(r)] = \alpha^r[\phi(\alpha E)f(r)]$ where $\phi(E)$ is a continuous function of the shift operator E, we have

$$\sum_{r=1}^{N_i}\log r\left(\frac{t}{a+t}\right)^r = \left[\left(\frac{t^r}{(a+t)^r}\left(\alpha(1+\Delta)-1\right)\log r\right]_1^{Ni+1}$$

$$= \left[\frac{t^r}{(a+t)^r}(\alpha-1)^{-1}\left(1+\frac{\alpha}{\alpha-1}\Delta\right)^{-1}\log r\right]_1^{Ni+1}$$

$$= \frac{t^r}{(a+t)^r}\left(-\frac{a+t}{a}\right)\left[1 - \frac{\frac{t}{a+t}}{\frac{t}{a+t}-1}\,\Delta\log r + O(\Delta^2 \log r)\right]_1^{N_i}$$

$$= \left[-\frac{1}{a}\,\frac{t^r}{(a+t)^{r-1}}\left(1+\frac{t}{a}\log\frac{r+1)}{r}\right)\right]_1^{N_i+1}$$

$$= \left[-\frac{1}{a}\,\frac{t^{N_i+1}}{(a+t)}\left(1+\frac{t}{a}\log\frac{N_i+2}{N_i}\right)+\frac{t}{a}\left(1+\frac{t}{a}\log_e 2\right)\right]$$

$$= \frac{t}{a}\left(1+\frac{t}{a}\log_e 2\right) - \frac{t}{a}\left(\frac{t}{a+t}\right)^{N_i}\left(1+\frac{t}{a}\log_e\frac{N_i+2}{N_i+1}\right). \qquad (5)$$

Putting (5) on the R.H.S. of (3‴)

$$E(r/i) \simeq \frac{k}{\Gamma(k)}\left(\frac{a}{a+t}\right)^{k-1}\left\{\left[\frac{t}{a}\left(1-\left(\frac{t}{a+t}\right)^{N_i}\right)\right]\right.$$

$$+ (k-1)\,\frac{t}{a}\left[\left(1+\frac{t}{a}\log_e 2\right) - \left(\frac{t}{a+t}\right)^{N_i}\left(1+\frac{t}{a}\log_e\frac{N_i+2}{N_i+1}\right)\right]$$

$$\left. + \frac{k-1}{2}\log_e\left(\frac{a}{a+t}\right)\right\}. \qquad (6)$$

Replacing t by t_+ and (k, a) by (k', a') we get precisely by the same argument

$$E(r/i-1) = \frac{k'}{\Gamma(k')}\left(\frac{a'}{a'+t+\tau}\right)^{k'-1}\left\{\left[\frac{t+\tau}{a'}\left(1-\left(\frac{t+\tau}{a'+t+\tau}\right)^{N_i}\right)\right]\right.$$

$$+ (k'-1)\,\frac{t+\tau}{a'}\left[\left(1+\frac{t+\tau}{a'}\log_e 2\right)\right.$$

$$\left. - \left(\frac{t+\tau}{a'+t+\tau}\right)^{N_i}\left(1+\frac{t+\tau}{a'}\log_e\frac{N_i+2}{N_i+1}\right)\right]$$

$$\left. + \frac{k'-1}{2}\log_e\frac{a'}{a'+t+\tau}\right\} \qquad (7)$$

on having estimates of *(k, a) (k', a')* and the information of t, and N_t (which are generally known), one can obtain the mean number of additional births for the class of women who had already / number of births; as well as for those who had *(i —* 1) number of births at the observational point.

Estimation of *(k, a)* and *(k', a')* from Incomplete Data

We note at the very outset that the information relating to the two groups under comparison are available upto i-th and (i — 1)-th order of births covered upto the point of observation measured from the date of consummation of marriage. In other words, our data remains incomplete and our problem lies in estimating the facility parameters *(k, a) (k',* a') which are pertinent for the open and unobserved reproductive period following *i* and (/— 1) number of births respectively for the two grouQs. We assume

$$E\{\lambda^{(0)}/i\} = \alpha_i + E(\lambda^{(i)}) \tag{8}$$

where {A^IO)//} represents the mean monthly fecundability rate upto z'-th order of pregnacy A^('> represents the mean fecundability rate from the i-th order of pregnancy upto for the end of the reproductive span. We have,

$$E\{\lambda^{(0)}/i\} = \alpha_i + \alpha_i \frac{k}{a} \tag{9}$$

$$\text{Var}\{\lambda^{(0)}/i\} = \frac{k}{a^2} \tag{10}$$

(9) and (10) ⇒ $\hat{\alpha}_i = E(\lambda^{(0)}/i) - \hat{a}\,\text{Var}(\lambda^{(0)}/i)$ (11)

and (10) ⇒ $\hat{k} = a^2\,\text{Var}(\lambda^{(0)}/i).$ (12)

Also from (10), the third refined moment (from the mean) of the p.d.f. of $\lambda^{(i)}$ is given by

$$\mu_3 = \frac{2k}{a^3} = \frac{2}{a}\text{Var}(\lambda^{(0)}/i) \tag{13}$$

$$\hat{a} = 2\,\text{Var}(\lambda^{(0)}/i)/\hat{\mu}_3. \tag{14}$$

From (11), it follows that

$$\hat{k} = 4\,(\text{Var}(\lambda^{(0)}/i))^3/\hat{\mu}_3^2 \tag{15}$$

and $\hat{\alpha}_i = [E(\lambda^{(0)}/i)] - 2\,\frac{[\text{Var}(\lambda^{(0)}/i)]^2}{\hat{\mu}_3}.$ (16)

Numerical Illustration of the Technique based on Hypothetical Data

Let the moments of some hypothetical fecundability distribution of $(\lambda^{(i)})$ obtained from Model sampling of size 100 from a Gamma population with parameters $k = 4.06$ and $a = 15.63$ (index parameter) which corresponds to a mean value of $\lambda^{(i)} = 0.26$ and variance as .016636 for $i = 3$ (conception of order three). These gave the sample moments

$$\bar{X}_i = 0.242;$$
$$S_2 i = 0.012963;$$
$$S_3 i = 0.0021288;$$

using (14), (15) and (16)

The estimated parameters are as follows :

$$\hat{a} = 2\,(.012963)/.0021288 = 12.178692$$
$$\hat{k} = 4\,(.012963)/(.0021288)^2 = 1.9333$$
$$\hat{\alpha}_i = .242 - 2\,(.012963)^2/.0021288 = 0.084.$$

The monthly fecundability rates for women with order of pregnancy upto $i = 3$ and for $i > 3$ were calculated as per the following scheme. We have

$$E\,\{\lambda^{(0)}/i\} = \alpha_i + E\,(\lambda^{(i)}) = p_{(i)}{}^1$$
$$\Rightarrow E(\lambda^{(i)} = E\,\{\lambda^{(0)}/i\} - \alpha_i)$$

The mean monthly fecundability rate at the j-th month for women with order of pregnancies upto i is given by

$$E\,[(\lambda^{(0)}/i)]\,[1 - E\,(\lambda^{(0)}/i)]^{j-1} \qquad j = 1, 2, 3, \ldots$$

This is based on the tacit assumption

$$q_{(i)} = e^{-\int_j^{j+1} (\lambda^{(0)}\,(\tau)/i)d\tau}$$

Replacing $(\lambda^{(0)}\,(\tau)/i)$ by its expectation[2] which is constant over time and neglecting powers of $E\,(\lambda^{(0)}/i)$ the result follows. Whereas the mean monthly fecundability rate of women having order of pregnancies upto 3rd order at the j-th month is

1. For detailed explanation vide the footnote No. 2.

2.
$$q_{(i)} = e^{-\int_j^{j+1} (\lambda^{(0)}\,(\tau)/i)d\tau} \simeq e^{-\int_j^{j+1} E(\lambda^{(0)}/i)\,d\tau} = e^{-E(\lambda^{(0)}/i)}$$
$$p_{(i)} = 1 - e^{-E(\lambda^{(0)}/i)} = 1 - [1 - E(\lambda^{(0)}/i) + O\,(E\,(\lambda^{(0)}/i))]$$
$$p_{(i)} \simeq E(\lambda^{(0)}/i).$$

$$p_{(i)}\, q_{(i)}^{j-1} \quad j = 1, 2, 3, \ldots$$

where $$q_{(i)} = 1 - p_{(i)}.$$

The rates so worked out are presented in Table 1.

TABLE 1—MEAN MONTHLY FECUNDABILITY RATE UPTO THIRD AND HIGHER THAN THIRD ORDER OF CONCEPTIONS

Month	*Mean fecundability rate upto 3rd order pregnancy*	*Mean fecundability rate for higher than the 3rd order of pregnancies*
1.	0.242	0.158
2.	0.183436	0.133036
3.	0.139044	0.112016
4.	0.105396	0.094318
5.	0.079890	0.079416
6.	0.060556	0.066868
7.	0.045902	0.056303
8.	0.034794	0.047407
9.	0.026374	0.039917
10.	0.019991	0.033610
11.	0.015153	0.028299
12.	0.011486	0.023828

Similarly the sample moments of the observed fecundability distribution ($\lambda^{(i-1)}$) obtained from Model sampling from a Gamma Population with parameters k = 4.55 and a = 16.67 which corresponds to a mean of 0.273 and variance of 0.1638 and $(i - 1) = 2$ are as follows :

$$\bar{X}\,(i - 1) = 0.273$$
$$S_2,\,(i - 1) = 0.01638$$
$$S_3,\,(i - 1) = 0.0027128$$
$$\hat{a}' = (.273)/(.01638) = 16.6667$$
$$\hat{k}' = 4\,(.01638)^3/(.0027128)^2 = 2.3987$$
$$\hat{\alpha}_{i-1} = .273 - 2(.01638)^2/.0027128 = .075.$$

With these estimates following the same technique as for $i = 3$, the monthly fecundability rates for order of pregnancies upto $(i - 1) = 2$ above $(i - 1)$ have been calculated from the hypothetical data of monthly fecundability of women upto $(i - 1)$-th order of parity. The result is shown in Table 2. Let us assume that the mean difference between the age of the mothers having $i = 3$ and $(i - 1) = 2$ pregnancies respectively is t years while having the same age at marriage for both the groups. We further assume that the residual reproductive span for mothers with already 3 pregnancies be $t = 20$ years, on the date of observation.

TABLE 2- MEAN FECUNDABILITY RATE UPTO THE SECOND AND HIGHER THAN THE SECOND ORDER OF CONCEPTION

Month	*Mean fecundability rate for women with second order ofpregnancy*	*Mean fecundability rate for women with order ofpregnancies exceeding two*
1.	0.0259	0.1838
2.	0.192919	0.150018
3.	0.142212	0-122444
4.	0.105379	0.099939
5-	0.078086	0.0S1570
6.	0.057862	0.066578
7.	0.C42875	0.054341
8.	0.031770	0.044353
9.	0.023542	0.036201
10.	0.017445	0.029547
11.	0.012926	0.024116
12.	0.009579	0.019684

Finally we take $N_i = 7$, the possible upper bound of the residual number of conceptions, given that the $i = 3$ (three) conceptions have already taken place. This gives from (6)

$$E(r/i) = \frac{k}{\Gamma(k)} \left(\frac{a}{a+t}\right)^{k-1} \left\{ \left[\frac{t}{a}\left(1 - \left(\frac{t}{a+t}\right)^{N_i}\right)\right] \right.$$
$$+ (k-1)\, \frac{t}{a}\left(1 + \frac{t}{a} \log_e 2\right)$$
$$- \left(\frac{t}{a+t}\right)^{N_i}\left(1 + \frac{t}{a} \log_e \frac{N_i + 2}{N_i + 1}\right)\Bigg]$$
$$\left. + \frac{k-1}{2} \log_e \frac{a}{a+1} \right\}$$

on substituting the estimated values of k, a (for $i = 3$) and the assumed value of t and N_i we get

$$[E(\hat{r}/i)] = 4.0828975.$$

Similarly using (7)

$$E(r/i-1) = \frac{k'}{\Gamma(k')}\left(\frac{a'}{a'+t+\tau}\right)^{k'-1}\left\{\left[\frac{t+\tau}{a'}\left(1-\left(\frac{t+\tau}{a'+t+\tau}\right)^{N_i}\right)\right]\right.$$
$$+(k'-1)\frac{t+\tau}{a'}\left[\left(1+\frac{t+\tau}{a'}\log_e 2\right)\right.$$
$$\left.-\left(\frac{t+\tau}{a'+t+\tau}\right)^{N_i}\left(1+\frac{t+\tau}{a'}\log_e\frac{N_i+2}{N_i+1}\right)\right].$$
$$\left.+\frac{k'-1}{2}\log_e\frac{a'}{a'+t+\tau}\right\}$$

On substitution of the estimated values of a', k' for i — 1 =2 and the assumed values of t, T and Ni we get

$$E(\hat{r}li - 1) = 3.890854.$$

Some Comments on the Numerical Results

A comparison of the analysis made on the basis of the Hypothetical data based on a model sampling from two Gamma populations reveal that mothers who had three conceptions earlier are exposed to the risk of four additional conception whereas the comparatively younger mothers who had two conceptions are also exposed to the same risk of about four conceptions. Although these findings are based on purely hypothetical data if true reveal that the conception rate for the younger cohort is declining on the average of one conception in the whole reproductive period. This may mean virtually a reduction of about 0.8 births (taking a mean ratio of live birth to conception as 0.87) on the average in the whole reproductive span. Since the exercise is purely methodological just to illustrate the applicability of the model to compare the future fertility status of two marriage cohorts having the mean age at marriage to be the same the limitations of the findings of the exercise has to be understood as such.

With more extensive live data, however the model is expected to be sensitive enough to show the major difference (if any) in the fertility level of two comparable marriage cohorts; only on the basis of available fertility records upto a short observable duration of marital exposure.

Some Possible Generalizations of the Model

One major limitation of the methodology may be with the appropriate validity of William Brass model in this context. William Brass model utilized here does not take account of the post partum Infecundable exposure following each pregnancy; as well the possibilities of a finite number of renewal of births with gradually decreasing intensities during the entire marital span.

While the post partum infecundable period can be incorporated in Brass Model using some generalized models (Biswas, 1973) a damping factor asymptotically decreasing the pregnancy rate to zero in the renewal process of reproduction (Biswas, 1981) would, on doubt, be of some kind of improvement to the present model even at the cost of considerable mathematical complications.

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